The Black-Scholes Formula

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Hedging

If you **writing a call option**, you are exposed to the risk of the stock price rising above the strike price at expiration

In order to **cover the call**, you need at expiration, cash equal to the difference between the current price of the stock and the strike price of the option (ST-K)

In Black-Scholes, the hedge is a **portfolio compounded with some shares of the underlying stock and some riskless assets** (US T-Bonds)

How much of each to put in the hedge is the key to determining the option's value.

Perfect hedge

A perfect hedge always pays exactly the amount necessary to cover the option.

Payoff replication:

- if the option expires in the money, the hedge provides exactly the amount necessary to cover the call
- if the option expires out of the money the hedge will be worth nothing

Dynamic hedging

Dynamic hedging is a **trading strategy which replicates the payoff of the option throughout its life** (creating a perfect hedge)

It has a fixed and known total cost

Dynamic hedging is a **process** for managing the risk of options and it is **important for explaining an option's value.**

Dynamic hedging

Black-Scholes hedging is determined by the **weights of the hedging portfolio**. Because the parameters of the formula are **constantly changing**, the hedging portfolio **must be constantly updated to reflect the new weights…**

The portfolio which doesn't reflects the current Black-Scholes weights is out of balance. The process to keeping the portfolio in balance is called ***rebalancing***

If the hedging strategy is performed correctly, the value of the portfolio will be equal to the value of the option at all times…

Costs of the hedging

1. **Set-up cost** (initial cash flows associated with the hedging strategy)
2. **Maintenance costs**: infusion and transaction costs (costs associated with rebalancing the hedge, e.g. buy more share of stock or bonds)

In addition, there are:

- **fees and taxes** associated with making the transactions
- costs resulting from the **bid-ask spread** and
- the **inability** to execute trades at exactly the price specified by the strategy

Self-Financing Dynamic Hedging

A hedging strategy whose total cost at any time (excluding transaction costs) is equal to set-up costs is called self-financing.

The coin-tossing example

A gambler decides making a bet of $100.000 He needs 3 faces to win

12.500→25.000
25.000→50.000
50.000→100.000

The Delta of an option

The Delta of an option is the rate of the change in the option price with respect to the change in the stock price

When we look at changes in the value of an option, we are interested in changes relative to the underlying stock price.

If the price of the stock changes $1, what will be the change in the option's value?

The Delta of an option

- **Option is deep in the money and near expiration: the delta is close to one**, so there is a dollar for dollar relationship between movements of the spot price and movements of the option's value
- **Option is deep out of the money and near expiration**: **the delta is close to zero** (because the option's value is absolutely insensitive to changes in the stock price) and it will be worth almost nothing

Implicit in the B&S model there is only factor of uncertainty in the value of an option: volatility

In truth, the uncertainty implicit in an option is more complicated than this.

The Delta of an option

Real world is more complicated: the dynamic of stock price is more complicated than a geometric brownian motion

Jump risk: if there is a probability of stock price will unexpectedly jump downward, this can change the option's value **even when the spot price does not change** (Brownian motion model does not include this effect)

In principle, we can observe this jump comparing the spot price and option value at some time t_0 , doing the same at a later time t_1

Because the price of the option is not controlled by the stock price alone, this may note give an accurate estimate of the delta of the option

Even if we ignore probabilities of jumps and changing volatility, the value of the options does change as time elapses (see parameter theta)

The Delta of an option

In the B&S world, **stock price are supposed to follow a GBM with constant volatility** and the option's value changes for only two reasons:

1. **Time change and**
2. **Stock price change**

We can use an approximation to the delta of the option: the actual rate of change in the value of the option with respect to the change in the stock price…

The Delta of an option

Example:

$S_0 = 1,87$; $c_0 = 0,059$

$S_1 = 1,89$; $c_1 = 0,06$

$$\Delta = \frac{C_{t_0} - e^{-r(t_1 - t_0)} C_{t_1}}{S_{t_0} - e^{-r(t_1 - t_0)} S_{t_1}} = \frac{0,06 - e^{0,047(1/365)} 0,059}{1,89 - e^{0,047(1/365)} 1,87} = \frac{0,001}{0,02} \cong 0,05$$

The value of the delta is 0,05

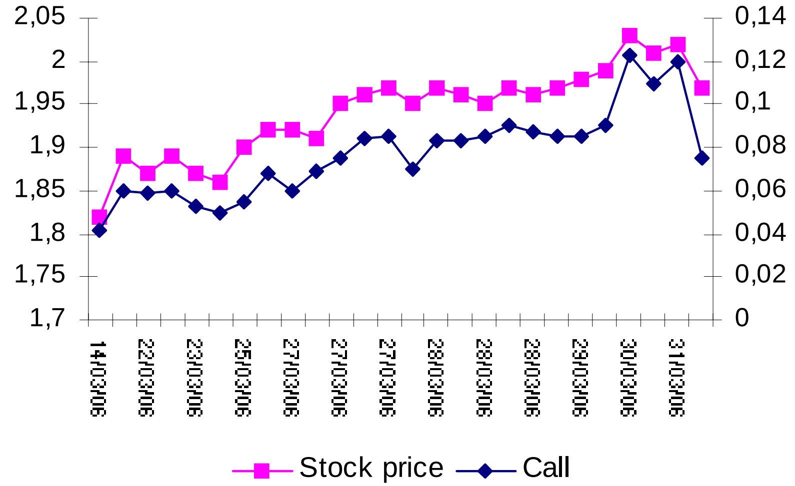
The numerator says the time adjusted change in the price of the option was \$ 0,02 while the time adjusted change in the stock price was \$0,001

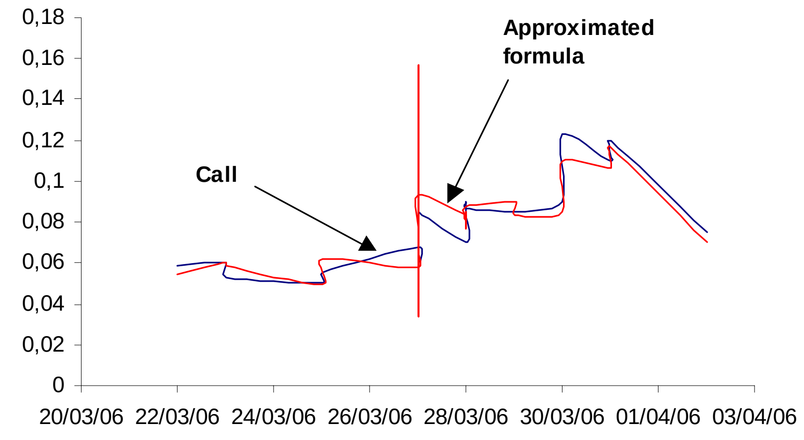
The ratio says that for every \$1 change in the stock price, there was an approximately \$0,05 in the value of the option

How the formula works in the real world?

Call evolution for Transener

Date	Stock price	Call	Days to maturity	Var S	Var Call	Delta	Call (formula)
14/03/06	1,82	0,042	38				
21/03/06	1,89	0,06	31	0,07	0,018	0,257	
22/03/06	1,87	0,059	30	-0,02	-0,001	0,050	0,055
23/03/06	1,89	0,06	29	0,02	0,001	0,050	0,060
23/03/06	1,87	0,053	29	-0,02	-0,007	0,350	0,059
25/03/06	1,86	0,05	27	-0,01	-0,003	0,300	0,050
25/03/06	1,9	0,055	27	0,04	0,005	0,125	0,062
27/03/06	1,92	0,068	25	0,02	0,013	0,650	0,058
27/03/06	1,9209	0,06	25	0,0009	-0,008	-8,889	0,069
27/03/06	1,91	0,069	25	-0,0109	0,009	-0,826	0,157
27/03/06	1,95	0,075	25	0,04	0,006	0,150	0,036
27/03/06	1,96	0,084	25	0,01	0,009	0,900	0,077
27/03/06	1,97	0,085	25	0,01	0,001	0,100	0,093
28/03/06	1,95	0,07	24	-0,02	-0,015	0,750	0,083
28/03/06	1,97	0,083	24	0,02	0,013	0,650	0,085
28/03/06	1,96	0,083	24	-0,01	0	0,000	0,077
28/03/06	1,95	0,085	24	-0,01	0,002	-0,200	0,083
28/03/06	1,97	0,09	24	0,02	0,005	0,250	0,081
28/03/06	1,96	0,087	24	-0,01	-0,003	0,300	0,088
29/03/06	1,97	0,085	23	0,01	-0,002	-0,200	0,090
29/03/06	1,98	0,085	23	0,01	0	0,000	0,083
30/03/06	1,99	0,09	22	0,01	0,005	0,500	0,085
30/03/06	2,03	0,123	22	0,04	0,033	0,825	0,110
31/03/06	2,01	0,11	21	-0,02	-0,013	0,650	0,107
31/03/06	2,02	0,12	21	0,01	0,01	1,000	0,117
02/04/06	1,97	0,075	19	-0,05	-0,045	0,900	0,070





The approximate formula

The formula is a good approximation because:

1. If we choose a **time t_1 very close to time t_0** and apply the formula, then we obtain values for the delta that are very close to the actual value of the delta at time t_0
2. Each time we substitute the approximate value of the delta for the real value, we produce a small error. But these small errors do not accumulate, and do not make our approximation useless!!

Delta – approximate formula

What makes the formula an approximation is the fact that we compare the change in the option price to the change in the stock price over small period of time.

If all times are infinitesimally close, we get a formula that is no longer approximate but is precisely correct...

What if the denominator of the formula is zero?

This does not invalidate the formula, because according to the BM model, the probability that the stock price at time t1 will be exactly equal to... is zero (the probability of any given “event”is always zero, only ranges of events matter in probability theory) Of course, in reality it can happen.

The Black-Scholes hedging strategy

If we **know the delta** of a european call or put option, **we can hedge the option and determine its theoretical value**

The assumptions behind the Black-Scholes formula play a key role in determining the value of an option. Identifying how relaxing them is an important part of understanding the Black-Scholes method

Hedging strategy in Black-Scholes

A very important part is to understand how the formula translates into a **hedging strategy that replicates the payoff of the option**

Then, **no-arbitrage assumptions ensures that the value of the option is equal to the cost of the hedging strategy**

Hedging strategy in Black-Scholes

The value of the hedging portfolio at time t, is the SAME as the B&S Formula for the option's value

$$\Delta_t S_t - B_t e^{-r(T-t)}$$

Once the hedging is set up, the next step is to maintain the hedge…

In order to keep the portfolio in balance, we will have to continuously monitor the values of Δ and B, buying or selling shares...

In order to ensure the strategy is self-rebalancing we want the rebalancing costs at every step to be as close to zero as possible

Hedging strategy in Black-Scholes

In reality, the hedging strategy is used to hedge an option not on a single share of stock but on some larger number of shares (In Argentina, a standard option contract is on 100 shares of the underlying)

Since delta is a number between 0 and 1, it represents the percentage of the lot of shares that should be purchased…

The hedging strategy is self-financing

In *BS world there is no transactions costs*, so

Set up costs=value of option=total cost of hedging

Therefore, set-up costs equal the total cost of hedging, which **implies that the maintenance cost of hedging must equal zero and the portfolio is self-financing**

If the BS formula holds, then the hedging strategy must be self-financing. It would be more convincing if we started with the hedging strategy and then showed that it replicates the payoff of the option and is self-financing

Rebalancing the portfolio

When we **set up the hedging portfolio at time t_0** its value is

$$C_T = \Delta_{t_0} S_{t_0} - B_{t_0} e^{-r(T-t_0)}$$

Suppose that at time t_1 we decide to **rebalance the portfolio**. The amount spent at t_0 and t_1 are

Time	**Amount spent**
t_0	$\Delta_{t0} S_{t0} - e^{-r(T-t0)} B_{t0}$
t_1	$(\Delta_{t1} - \Delta_{t0}) S_{t0} - e^{-r(T-t0)} (B_{t1} - B_{t0})$

Because **we need to compare the value of all purchases at t_0**, the money spent at time t_1 must be discounted by a factor of $e^{-r(t1-t0)}$.

Rebalancing the portfolio

The **money spent on hedging at time t$_1$** is

$$e^{-r(t_1-t_0)}(\Delta_{t_1}-\Delta_{t_0})S_{t_1}-(B_{t_1}-B_{t_0})e^{-r(T-t_0)}$$

$$e^{-r(t_1-t_0)}e^{-r(T-t_1)}(B_{t_1}-B_{t_0})=e^{-r(T-t_0)}(B_{t_1}-B_{t_0})$$

And the **money spent in setting up the hedge at time t$_0$ and rebalancing at time t$_1$** is

$$(\Delta_{t_0}S_{t_0}-B_{t_0}e^{-r(T-t_0)})+e^{-r(t_1-t_0)}(\Delta_{t_1}-\Delta_{t_0})S_{t_1}-(B_{t_1}-B_{t_0})e^{-r(T-t_0)}$$

Rearranging this a bit, we obtain

$$\Delta_{t_0}(S_{t_0}-e^{-r(t_1-t_0)}S_{t_1})+e^{-r(t_1-t_0)}(\Delta_{t_1}S_{t_1}-e^{-r(T-t_1)}B_{t_1})$$

Replacing Δt_0 in the last expression for the approximate delta, we have

$$\frac{C_{t_0}-e^{-r(t_1-t_0)}C_{t_1}}{S_{t_0}-e^{-r(t_1-t_0)}S_{t_1}}(S_{t_0}-e^{-r(t_1-t_0)}S_{t_1})+e^{-r(t_1-t_0)}(\Delta_{t_1}S_{t_1}-e^{-r(T-t_1)}B_{t_1})$$

Rebalancing the portfolio

$$\frac{C_{t_0} - e^{-r(t_1-t_0)}C_{t_1}}{S_{t_0} - e^{-r(t_1-t_0)}S_{t_1}}(S_{t_0} - e^{-r(t_1-t_0)}S_{t_1}) + e^{-r(t_1-t_0)}(\Delta_{t_1}S_{t_1} - e^{-r(T-t_1)}B_{t_1})$$

Option's value at time t_0

Option's value at time t_1 (adjusted for time value)

B&S formula for the call at time t_1 adjusted for time value

$$(C_{t_0} - e^{-r(t_1-t_0)}C_{t_1}) + e^{-r(t_1-t_0)}(\Delta_{t_1}S_{t_1} - e^{-r(T-t_1)}B_{t_1}) = C_{t_0}$$

This says **if we set up the portfolio at t_0 and rebalance at t_1, the total cost of the option is the cost of the option at time t_0**

If BS holds at time t_0 , it holds at time N-1; the same arguments imply that it holds at time t_{N-2}, t_{N-3}….**so BS formula must hold at time t_0**

The Black-Scholes formula for Δ and B

The formulas are given in terms of cumulative normal distribution function:

$$d_1 = \frac{\ln\left(\frac{S}{X}\right) + r_f T}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$$C = SN(d_1) - Xe^{-rT} N(d2)$$

Black-Scholes and the Brownian Motion Model

Computing **the probability that C expires in the money** is equivalent to computing the probability that $S_T \geq K$

The return on S from time t to time T is given by $\ln\left(\frac{S_T}{S_t}\right)$

This return is a random variable with mean $\left(r - \frac{\sigma^2}{2}\right)(T - t)$

And standard deviation $\sigma\sqrt{T - t}$

Therefore, the random variable is given by $\dfrac{\ln\left(\frac{S_T}{S_t}\right) - \left(r - \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}$

This is normally distributed, with mean zero and standard deviation 1

Black-Scholes and the Brownian Motion Model

Now, we transform the equation ST≥K dividing both sides by **St**, taking logarithms, substracting **r - σ² / 2 (T-t)** and finally, by dividing by $\sigma\sqrt{T - t}$

$$\frac{\ln\left(\frac{S_T}{S_t}\right)-\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \geq \frac{\ln\left(\frac{K}{S_t}\right)-\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Because we want an expression with "less than or equal" (to use the cumulative normal distribution) **we negate this equation and obtain:**

$$\frac{\ln\left(\frac{S_t}{S_T}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \leq \frac{\ln\left(\frac{S_t}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Black-Scholes and the Brownian Motion Model

For the original statement to be true (ST≥K) then the random variable on the left-hand side of equation must be less than or equal to the value on the right-hand side

$$\frac{\ln\left(\frac{S_t}{S_T}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \leq \frac{\ln\left(\frac{S_t}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

The probability the inequality holds is exactly given by the cumulative normal distribution of the right-hand side, so the probability that C will expire in the money is

$$P(S_T \leq K) = N\left(\frac{\ln\left(\frac{S_t}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

Excercises

1. Using B&S formula, calculate the fair value for a call option on Petrobras, for different strike prices. Today is 7/4/06 and the annual risk free rate is 4,9%. Then compare this value with the option market value.

2. Calculate the implied volatility for K=4

PBEC3.20AB	0,195	-7,14	0,21	0,19	0,195	0,19	6	19.780	13:21
PBEC3.20JU	0,315	+ 1,61	0,31	0,32	0,32	0,31	6	18.762	13:50
PBEC3.40AB	0,055	0,00	0,055	0,04	0,055	0,04	6	2.310	13:42
PBEC3.40JU	0,181	-2,69	0,186	0,181	0,181	0,181	1	1.755	13:38
PBEC3.60AB	0,008	-20,00	0,01	0,009	0,009	0,008	3	2.975	13:53
PBEC3.60JU	0,101	+ 1,00	0,10	0,10	0,105	0,10	7	7.948	13:46
PBEC3.80AB	0,003	-40,00	0,005	0,003	0,003	0,003	14	717	13:53
PBEC3.80JU	0,05	0,00	0,05	0,053	0,053	0,05	2	14.750	13:58
PBEC4.00AB	0,001	-50,00	0,002	0,002	0,002	0,001	4	199	13:33
PBEC4.40AB	0,001	0,00	0,001	0,001	0,001	0,001	1	60	12:55

Option market value

Petrobras	3,38	3,37	3,36	3,40	+0,30	2.228.858	14:41

Stock price