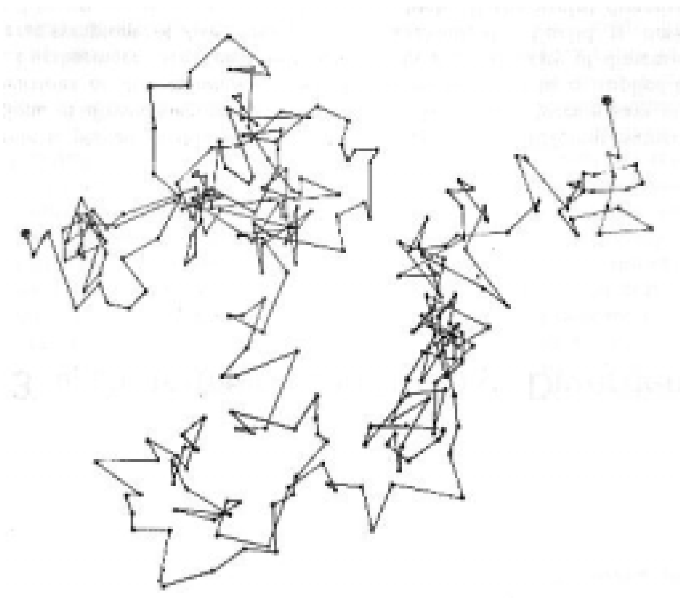
Geometric Brownian Motion

Dr. Guillermo López Dumrauf
dumrauf@fibertel.com.ar

Brownian Motion: the history

The model has its origins in a physical description about the **motion of a heavy particle suspended in a medium of light particles** (Robert Brown,1827) which can be easily observed in a microscope



Albert Einstein (1905) succeeded in stating the mathematical laws governing the movements of particles

M.F.M. Osborne (1959) was a physicist who applied the concept to the stock market

Browanian Motion was then more generally accepted because it could now be treated as a practical mathematical model

Brownian motion - description

Implicit in the GBM model is the concept that prices follow a "random walk"

A "random walk" is essentially a Brownian Motion where future **price movements are determined by present conditions alone** and are independent of past movements...

Around 1900, Louis Bachelier first proposed that financial markets follow a "random walk" which can be modeled by standard probability calculus

Samuelson (1965) proved, expanded, and refined Bachelier's discovery

Brownian motion - description

The light particles move around rapidly, occasionally randomly crash into the heavy particle. Each collision slightly displaces the heavy particle, but **the direction and magnitude of this displacement is random and independent from all the other collisions, but the nature of this randomness does not change from collision to collision** (each collision is independent, identically distributed random event)

GBM takes this situation, and using some mathematics, derives that the displacement of the particle over a longer period of time must be normally distributed with a **mean and standard deviation depending only on the amount of time that has passed**

If the particle's movement is very volatile over the short run, it will be **proportionally volatile over the long run…**

Brownian motion and financial assets - analogy

Imagine **prices as heavy particles** that are jarred around by **lighter particles, trades**. Each trade moves the price slightly

Because prices change in proportion to their size, then a better comparison is the **expected percentage change in the stock price** (the percentage change is the same regardless of its value)

GBM is better adapted to model stock returns than absolute price changes. Theories based on percentage changes are called geometric. Conversely, theories based on absolute changes are called arithmetic

GBM assumptions

GBM describes the **probability distribution of the future price of a stock**. The basic assumption of the model is as follows:

- **The return on a stock price between now and some very short time later (T-t) is normally distributed**
- **The standard deviation of this distribution can be estimated from historical data**
- **ST volatility is a good predictor of the LT volatility**

The stock price models employed in option pricing **are not predictive but probabilistic**: they assume a distribution of future prices derived from historical data and current market conditions

GBM – mean and standard deviation

The mean of the distribution is μ times the amount of time μ(T-t) **(The expected rate of return changes in proportion to time)**

$$\left(\mu - \frac{\sigma^2}{2}\right)(T - t)$$

Where: μ = instantaneous expected return σ = instantaneous standard deviation

It says that short term returns alone are not a good predictor of long term returns. Volatility tends to depress the expected returns below what the short term returns suggest (**the *average* amount the stochastic component depresses returns in a single move is $\sigma^2 / 2$)**

The standard deviation of return increases in proportion to the square root of the amount of time (Bachelier, 1900)

$$\sigma\sqrt{T - t}$$

Volatility paths

Volatility measure: standard deviation of short term returns on the stock (we can measure how much the daily return on the stock deviates from its average daily return over some period of time, e.g. three months)

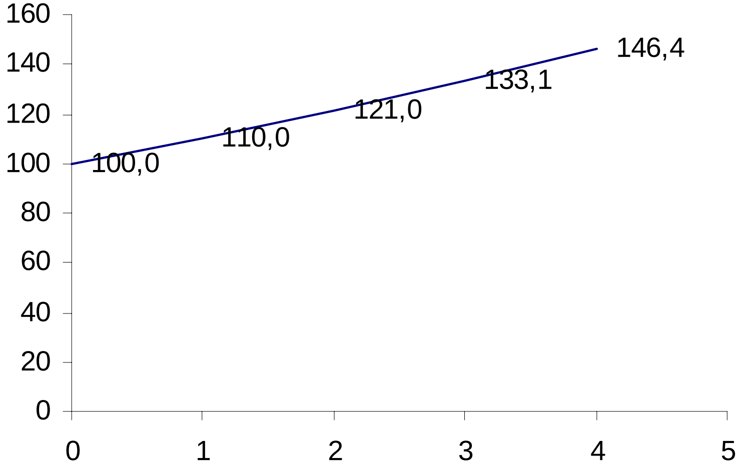
GBM examines the relation between the long and short term price behavior of the stock and states that the ST volatility is a perfect predictor of the LT volatility

The model only discusses what happens in a very short period of time

Volatiltity jumps

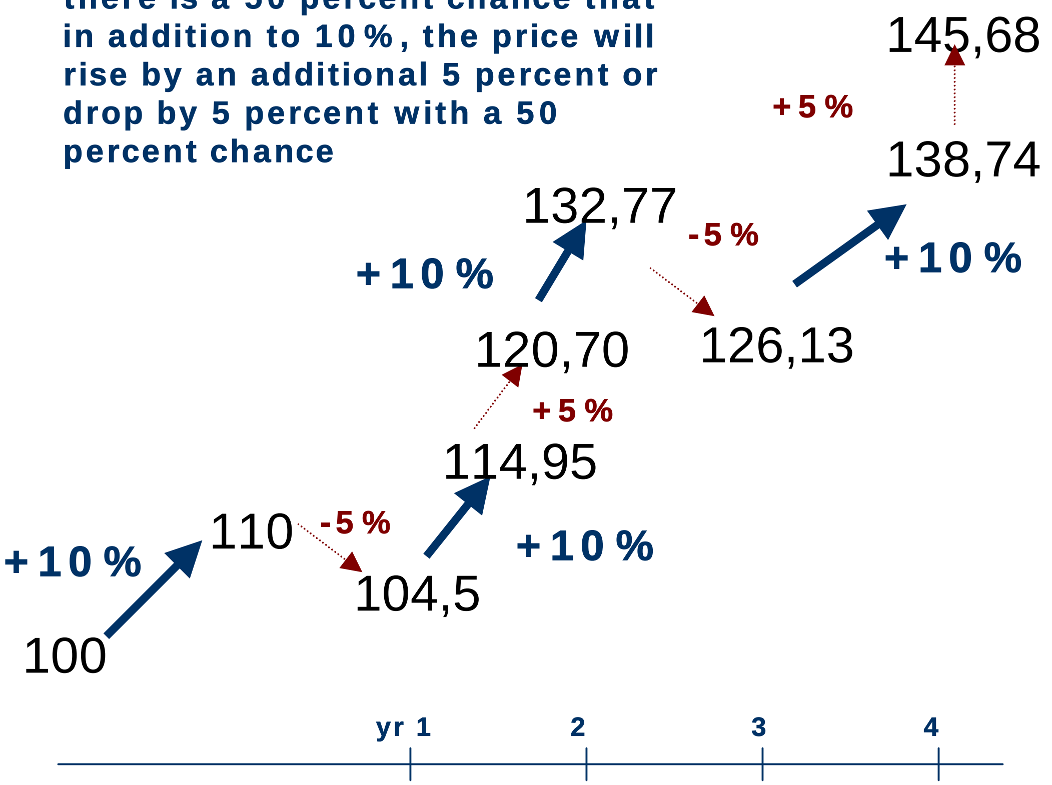
No Volatility

Suppose a bank deposit of $100 that earn an annual interest rate of 10% over four



Adding Volatility

Suppose at the end of each year there is a 50 percent chance that in addition to 10%, the price will rise by an additional 5 percent or drop by 5 percent with a 50 percent chance



The volatility jumps should not canceled themselves: 100(1,1)(1-0,05)(1,05)=114,95…

Volatility jumps and return

A positive return followed by an equal (but opposite) negative return **results in a slightly negative return!**

1,05 x 0,95=0,9975

$$(1+x)(1-x)=1-x^2$$

Stock prices and its relation with BM

Prices changes can be decomposed into two components:

- **Deterministic component** (the guaranteed 10 percent compounding each year)
- **Stochastic or random component** (the plus or minus 5 percent "jump" experienced each year on top of the guaranteed 10%)

The stochastic component is normally distributed with an expected value of zero (is symmetric about zero, just as the random "jump" was symmetric about zero)

The standard deviation of the stochastic component controls "how much" volatility there is on top of the deterministic component

Stock prices and its relation with BM

The long term returns on a stock are proportional to

$\mu - \sigma^2 / 2$ (and not to μ)

Because X^2 itself represent the result of two price moves, the *average* amount the stochastic component depresses returns in a single move is $\sigma^2 / 2$

If a random variable representing the stochastic component of Brownian Motion, then we have

$$VAR(X) = E(X^2) - E(X)^2 = E(X^2) = \sigma^2$$

Example

μ=10% per annum

σ=35% per annum

The model predicts that the five-year returns are normally distributed,

With mean $$\left(0{,}10-\frac{0{,}20^2}{2}\right)5=19{,}37\%$$

And standard deviation $$35\%\sqrt{5}=78{,}2\%$$

If we observed the average one-day returns and the standard deviation of one-day returns, respectively, we would find that they are approximately

10% x (1/365) and $35\%\ x\ 1/\sqrt{365}$

GBM and the real world

GBM states that the mean and standard deviation of a stock are constant. Clearly this is not the case with the rate of return on a stock (in fact, the rate of return can not observed directly)

Fortunately, **only the instantaneous standard deviation is important for option pricing**...

and **standard deviation is less difficult to predict** if we compare the change in the stock price over small period of time, infinitesimally close...

Ex

deterministic part

stochastic part

$$\frac{\delta / S}{S} = \mu(\delta t) + \sigma\varepsilon\sqrt{\delta t}$$

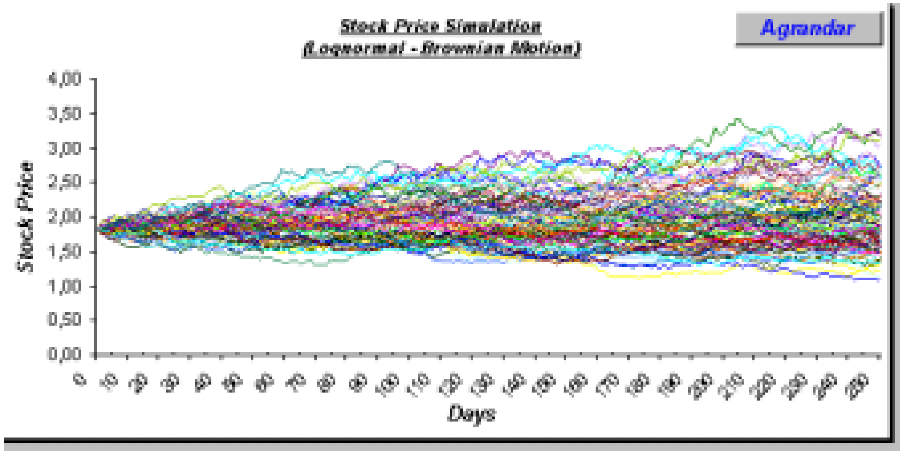
μ is a drift term or growth parameter that increases at a factor of time steps δt

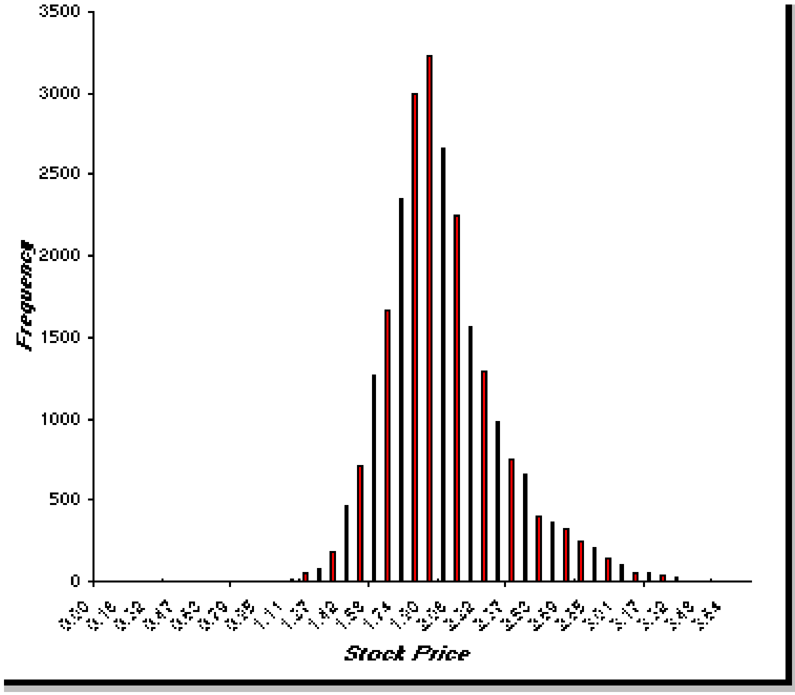
σ is the volatility parameter, growing at a rate of the square root of time, and ε **is a simulated variable**, usually following a normal distribution with a mean of zero and a variance of one

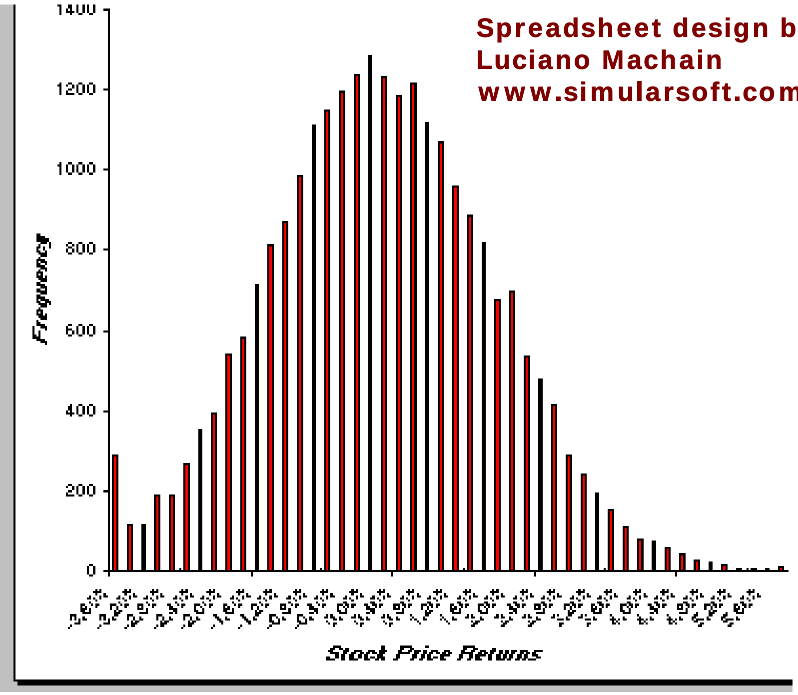
GBM as a tool for studying the stock markets

Now, we calibrate the model by computing its parameters over very short time intervals, and then using the conclusions to **infer information about the long–term returns and volatility…**

Stock Price	1,82
Mean	20,00%
Sigma	25,00%
# Simulations	100







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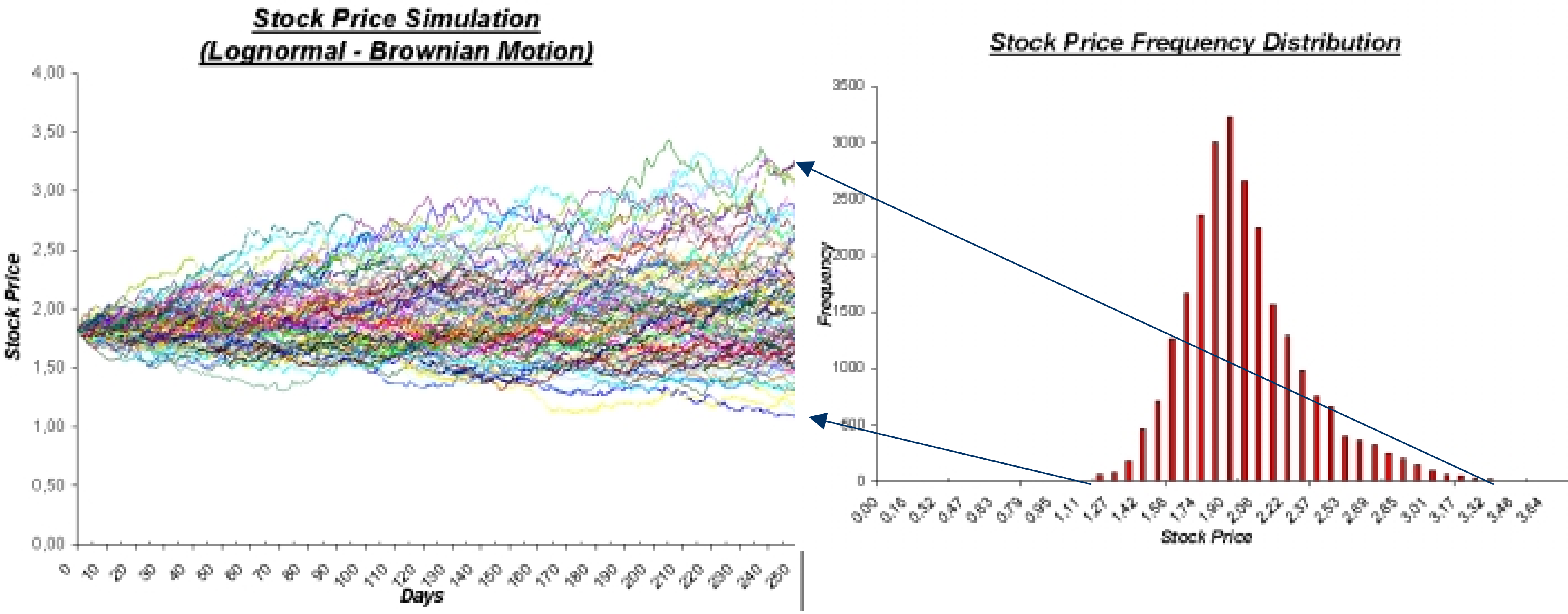
GBM as a tool for studying the stock markets



Stock price returns frequency distribution

GBM – empirical evidence

GBM state stock returns are normally distributed, and should be proportional to elapsed time and standard deviation should be proportional to the square root of elapsed time.

What does the data say?

- **Large movements in stock prices are more likely than GBM predicts** (leptokurtosis: the likelihood of returns near the mean and of large returns is greater than GBM predicts, while other returns tend to be less likely)
- **Downward jumps three standard deviations from the mean is three times more likely than a normal distribution would predict** (the theory underestimates the likelihood of large downward jumps)
- **Monthly and quarterly volatilities are higher than annual volatility and, daily volatilities are lower than annual volatilities** (Turner and Weigel,1990)

Steps in computing volatility

1. Fix a standard **time period** in terms of years (1/252)
2. Collect **price data** on the stock for each time period
3. Compute daily **returns** from the beginning to the end of each period in this way $r=\log(S_{t+1}/S_t)$
4. Compute the **average** value of the sample returns $r=(r_0+r_1+\ldots+r_n)/(N+1)$
5. Compute the **standard deviation** using the formula

$$\sigma = \frac{1}{\sqrt{\Delta t}}\sqrt{\frac{\sum_{j=1}^{n}(r_j-\bar{r})^2}{N}}$$

Day	Price	Return	Mean-squared differences
1	50		
2	50,79	1,57%	0,000397575
3	49,78	-2,01%	0,00025038
4	49,12	-1,33%	8,2523E-05
5	48,67	-0,92%	2,44101E-05
6	48,94	0,55%	9,59427E-05
7	48,69	-0,51%	7,37177E-07
8	49,14	0,92%	0,000181239
9	49,3	0,33%	5,64529E-05
10	48,68	-1,27%	7,04427E-05
11	48,78	0,21%	3,98782E-05
12	48,33	-0,93%	2,50511E-05
13	47,97	-0,75%	1,0329E-05
14	48,83	1,78%	0,000485403
15	47,82	-2,09%	0,000276827
16	46,62	-2,54%	0,000447387
17	46,93	0,66%	0,000118599
18	46,02	-1,96%	0,000234649
19	45,95	-0,15%	7,51068E-06
20	46,11	0,35%	5,9889E-05
		-0,43%	0,002865224

Variance	0,000159179
Std deviation	0,01261662
Std deviation (annual)	**24,104%**

Computing volatility

Closing prices for 20 consecutive days of a stock with μ =8% and σ=20%

Day	Price	Return	Mean-squared differences
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2	50,79	1,57%	0,000397575
3	49,78	-2,01%	0,00025038
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Variance	0,000159179
Std deviation	0,01261662
Std deviation (annual)	**24,104%**

Our computation gives a volatility of 23,47% percent. What happened?

Computing volatility

The main reason of discrepancy being that we computed the standard deviation of daily returns, not the instantaneous standard deviation of returns. Many independent tests would yield an average estimated volatility much closer to the real volatility of 20 percent

	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Trial 6
1	50	50	50	50,0000	50	50
2	50,358835	49,65485414	49,18601482	50,0110	49,3701364	50,52866336
3	50,799784	49,5573354	49,16695646	50,0219	48,96092276	50,95370014
4	51,372342	50,22815824	48,80934196	50,0329	48,7192867	50,36221671
5	52,730331	50,57378505	49,13443553	50,0439	49,35115384	50,02215797
6	51,987877	49,74555889	49,49093928	50,0548	50,10003219	50,42969546
7	52,912673	48,83404419	49,09614996	50,0658	49,84907514	50,00183434
8	53,33093	47,91681705	47,74421832	50,0768	49,55384587	50,09863072
9	52,73896	48,15641504	46,88934897	50,0877	49,01922753	50,71886054
10	52,869022	47,45130639	47,32425241	50,0987	48,69974165	50,43587589
11	52,383066	47,57334692	48,09485373	50,1097	48,42285069	49,78611168
12	52,565111	47,82658764	47,27032704	50,1207	48,94213163	48,805819
13	52,393156	47,727747	47,52346597	50,1317	49,28898942	48,79201774
14	52,641094	47,32346167	48,39103921	50,1427	48,53167958	48,65321998
15	53,082873	46,95258582	49,07004542	50,1537	49,44083634	48,50564207
16	53,550276	46,4183152	48,45755042	50,1647	50,06509644	48,7008461
17	54,169201	46,5852587	49,20033044	50,1757	49,4885997	48,67506076
18	56,007535	46,34418901	49,51273911	50,1866	50,70856764	48,18498127
19	56,486146	46,18891678	49,44888528	50,1976	51,15592362	48,3963617
20	56,376944	46,25976636	50,32444335	50,2087	50,9417566	49,08187469
Std deviation	**33,96**	**27,77**	**18,29**	**1,24**	**14,92**	**17,21**
Average Std Deviation		**20,31**				

The distribution of stock prices

GBM model concludes that stock returns are normally distributed and the stock prices are lognormally distributed

The annualized return is given by

$$\frac{1}{T-t_0}\ln\left(\frac{S_T}{S_{t_0}}\right)$$

The above equation is equal to

$$\frac{1}{T-t_0}\ln S_T - \frac{1}{T-t_0}\ln S_{t_0}$$

Let’s write X for this random variable

$$X = \frac{1}{T-t_0}\ln S_T - \frac{1}{T-t_0}\ln S_{t_0}$$

Rearranging a bit, we have a new random variable: X (the return, a random variable, normally distributed) plus a constant (not random)

$$X + \frac{1}{T-t_0}\ln S_{t_0} = \frac{1}{T-t_0}\ln S_T$$

Therefore, the natural logarithm of the future stock price (ln S_T) is normally distributed…

$$X(T-t_0) + \ln S_{t_0} = \ln S_T$$

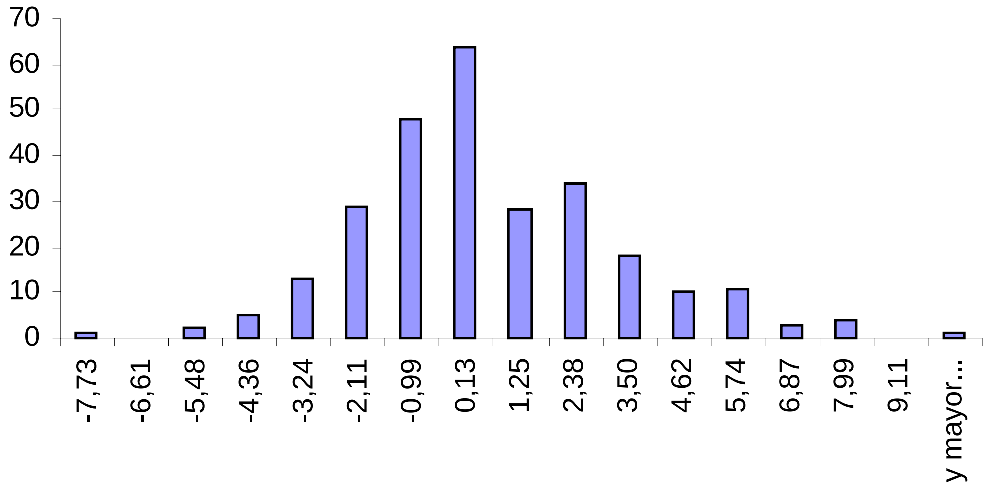
Exercises

1. Suppose we have a stock following a GBM with short term returns and volatility. You have to compute the probability of change of absolute value **greater than a fixed percent**

Stock:	Transener	
Annual Volatility	22%	(computed from daily closing prices for 40 days)
$\upsilon =$	15%	
$\upsilon - \sigma 2/2 =$	12,6%	

=DISTR.NORM(-A8;B4;B2;1)+1-DISTR.NORM(A8;B4;B2;1)

Change	Probability
2,5%	92,26%
5,0%	84,58%
10,0%	69,69%
20,0%	43,37%
50,0%	4,46%



Exercises

2. Collecting data from Economatica®, calculate the standard annual deviation between 31-12-04 to 31-12-05, and simulate a GBM for the price of Acindar using a drift of 10%. Then, comparing the stock price distribution following a GBM with the actual stock price distribution between 31-12-05 to 31-12-06

GBM simulation with Excel ®

Drift	0,1000
σ	0,3960
So	5,5200
Δt	0,0040
T	252

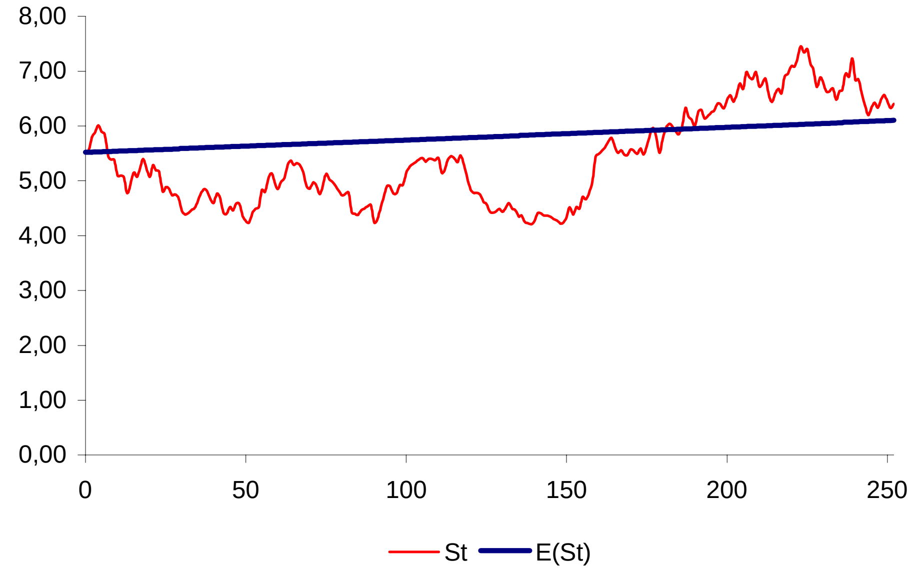
Data

=DISTR.NORM.INV(ALEATORIO(); 0; RAIZ(C4))

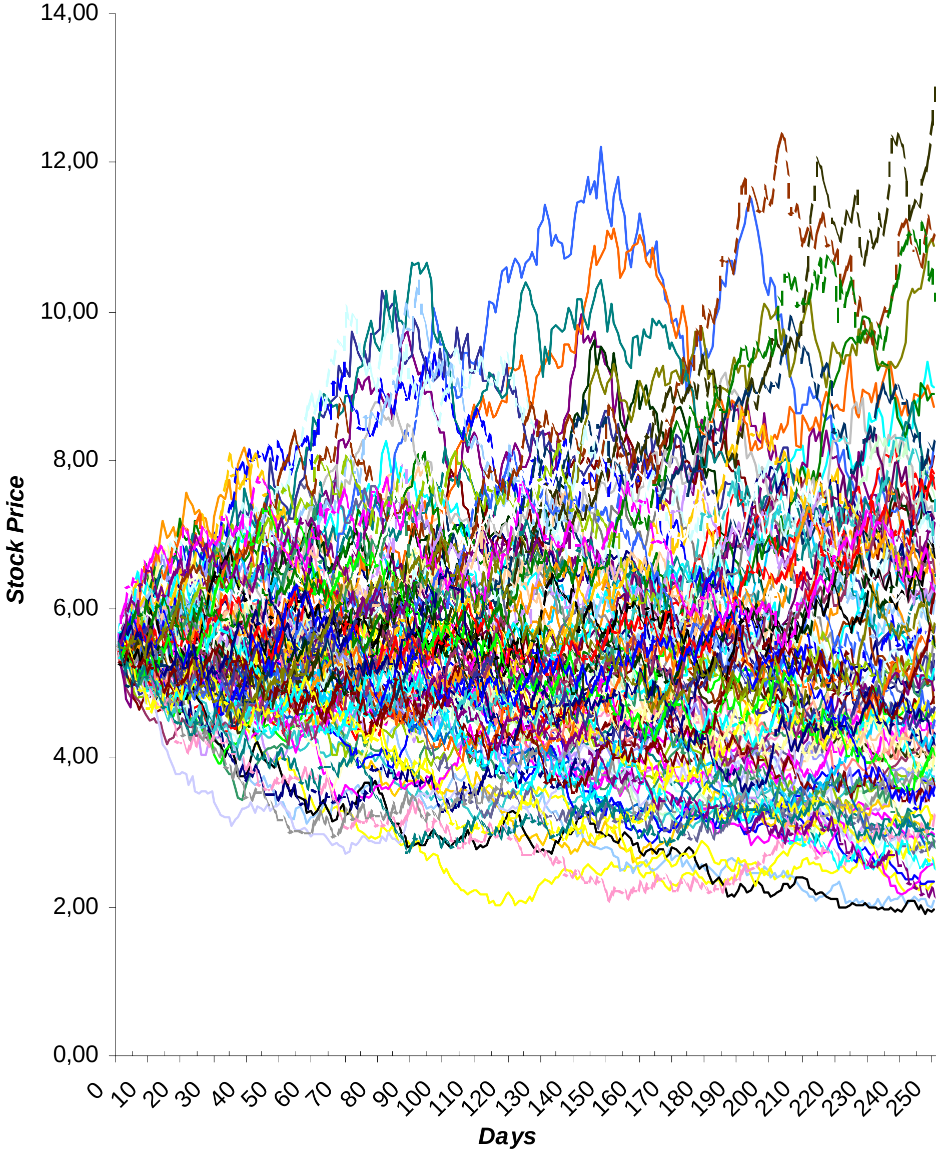
Expected value (with Drift)

Brownian Motion

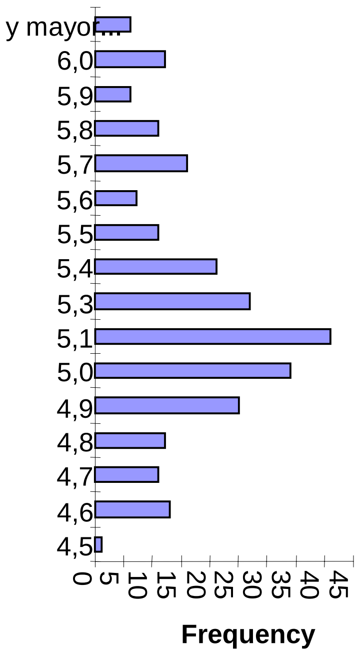
t	ΔW	W(t)	S(t)	E(S(t))	Var(S(t))
0,0000	********	0,0000	5,5200	5,5200	0,0000
0,0040	0,0167	0,0167	5,5571	5,5222	0,0190
0,0079	0,1063	0,1230	5,7965	5,5244	0,0380
0,0119	0,0356	0,1586	5,8792	5,5266	0,0571
0,0159	0,0555	0,2141	6,0105	5,5288	0,0762
0,0198	-0,0490	0,1651	5,8954	5,5310	0,0953
0,0238	-0,0225	0,1426	5,8436	5,5332	0,1145
0,0278	-0,1789	-0,0364	5,4443	5,5354	0,1338
0,0317	-0,0290	-0,0653	5,3827	5,5376	0,1530
0,0357	-0,0013	-0,0667	5,3803	5,5397	0,1724
0,0397	-0,1372	-0,2039	5,0962	5,5419	0,1917
0,0437	0,0006	-0,2033	5,0978	5,5441	0,2111
0,0476	-0,0143	-0,2176	5,0694	5,5463	0,2306
0,0516	-0,1506	-0,3682	4,7764	5,5485	0,2501
0,0556	0,0779	-0,2903	4,9265	5,5508	0,2696
0,0595	0,1086	-0,1817	5,1434	5,5530	0,2892
0,0635	-0,0358	-0,2175	5,0715	5,5552	0,3088



Acindar



14,00
12,00
10,00
8,00
6,00
4,00
2,00
0,00
Stock Price
0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 150 160 170 180 190 200 210 220 230 240 250
Days



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6,0
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4,6
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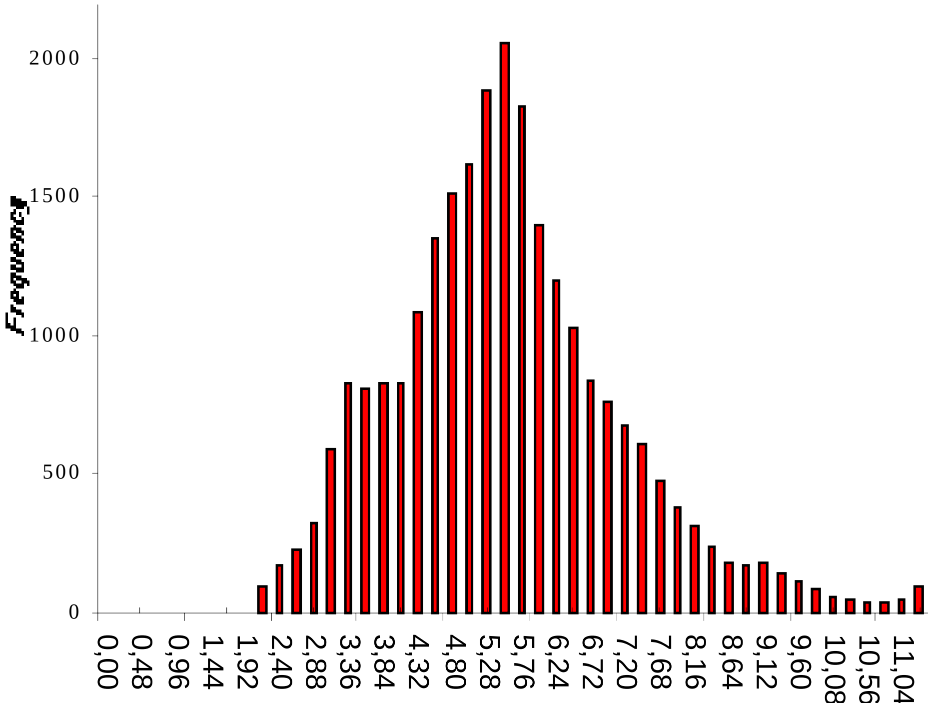
Clase
0 5 10 15 20 25 30 35 40 45
Frequency

Acindar

Simulated Distribution



Distribution ex post